2.2 Uniform Circular Motion

SECTION EXPECTATIONS

- Analyze, predict, and explain uniform circular motion.
- Explain forces involved in uniform circular motion in horizontal and vertical planes.
- Investigate relationships between period and frequency of an object in uniform circular motion.

KEY TERMS

- uniform circular motion
- centripetal acceleration
- centripetal force

Have you ever ridden on the Round Up at the Canadian National Exhibition, the ride shown in the photograph? From a distance, it might not look exciting, but the sensations could surprise you.

Everyone lines up around the outer edge and the ride slowly begins to turn. Not very exciting yet, but soon, the ride is spinning quite fast and you feel as though you are being pressed tightly against the wall. The rotations begin to make you feel disoriented and your stomach starts to feel a little queasy. Then, suddenly, the floor drops away, but you stay helplessly "stuck" to the wall. Just as you realize that you are not going to fall, the entire ride begins to tilt. At one point during each rotation, you find yourself looking toward the ground, which is almost directly in front of you. You do not feel as though you are going to fall, though, because you are literally stuck to the wall.

Figure 2.5 If this ride stopped turning, the people would start to fall. What feature of circular motion prevents people from falling when the ride is in motion and they are facing the ground?

What is unique about moving in a circle that allows you to apparently defy gravity? What causes people on the Round Up to stick to the wall? As you study this section, you will be able to answer these questions and many more.

Centripetal Acceleration

Amusement park rides are only one of a very large number of examples of circular motion. Motors, generators, vehicle wheels, fans, air in a tornado or hurricane, or a car going around a curve are other examples of circular motion. When an object is moving in a circle and its speed — the magnitude of its velocity — is

constant, it is said to be moving with **uniform circular motion**. The direction of the object's velocity is always tangent to the circle. Since the direction of the motion is always changing, the object is always accelerating.

Figure 2.6 shows the how the velocity of the object changes when it is undergoing uniform circular motion. As an object moves from point P to point Q, its velocity changes from $\vec{v}_1$ to $\vec{v}_2$. Since the direction of the acceleration is the same as the direction of the *change* in the velocity, you need to find $\Delta\vec{v}$ or $\vec{v}_2 - \vec{v}_1$. Vectors and $\vec{v}_1$ and $\vec{v}_2$ are subtracted graphically under the circle. To develop an equation for centripetal acceleration, you will first need to show that the triangle OPQ is similar to the triangle formed by the velocity vectors, as shown in the following points.

- $r_1 = r_2$ because they are radii of the same circle. Therefore, triangle OPQ is an isosceles triangle.
- $|\vec{v}_1| = |\vec{v}_2|$ because the speed is constant. Therefore, the triangle formed by $-\vec{v}_1$, $\vec{v}_2$, and $\Delta\vec{v}$ is an isosceles triangle.
- $r_1 \perp \vec{v}_1$ and $r_2 \perp \vec{v}_2$ because the radius of a circle is perpendicular to the tangent to the point where the radius contacts the circle.
- $\theta_r = \theta_v$ because the angle between corresponding members of sets of perpendicular lines are equal.
- Since the angles between the equal sides of two isosceles triangles are equal, the triangles are similar.

Now use the two similar triangles to find the magnitude of the acceleration. Since the derivation involves only magnitudes, omit vector notations.

- The ratios of the corresponding sides of similar triangles are equal. There is no need to distinguish between the sides r_1 and r_2 or v_1 and v_2, because the radii are equal and the magnitudes of the velocities are equal.

$$\frac{\Delta r}{r} = \frac{\Delta v}{v}$$

- The object travelled from point P to point Q in the time interval Δt. Therefore, the magnitude of the object's displacement along the arc from P to Q is

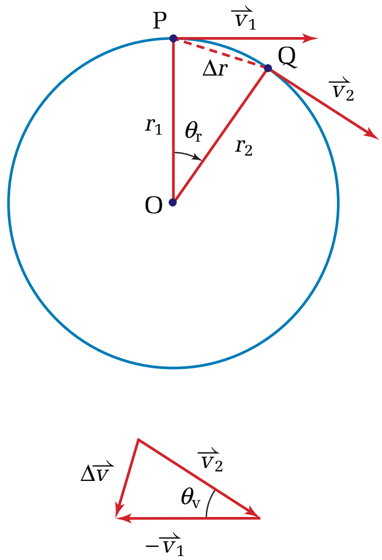
$$\Delta d = v\Delta t$$

- The length of the arc from point P to point Q is almost equal to Δr. As the angle becomes very small, the lengths become more nearly identical.

$$\Delta r = v\Delta t$$

- Substitute this value of Δr into the first equation.

$$\frac{v\Delta t}{r} = \frac{\Delta v}{v}$$



$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$

Figure 2.6 The direction of the change in velocity is found by defining the vector $-\vec{v}_1$ and then adding $\vec{v}_2$ and $-\vec{v}_1$. Place the tail of $-\vec{v}_1$ at the tip of $\vec{v}_2$ and draw the resultant vector, $\Delta\vec{v}$, from the tail of $\vec{v}_2$ to the tip of $-\vec{v}_1$.

MATH LINK

Mathematicians have developed a unique system for defining components of vectors such as force, acceleration, and velocity for movement on curved paths, even when the magnitude of the velocity is changing. Any curve can be treated as an arc of a circle. So, instead of using the *x*- and *y*-components of the typical Cartesian coordinate system, the vectors are divided into tangential and radial components. The tangential component is the component of the vector that is tangent to the curved path at the point at which the object is momentarily located. The radial component is perpendicular to the path and points to the centre of the circle defined by the arc or curved section of the path. Radial components are the same as centripetal components.

- Divide both sides of the equation by Δt. $\quad \dfrac{v}{r} = \dfrac{\Delta v}{v\Delta t}$
- Recall the definition of acceleration. $\quad a = \dfrac{\Delta v}{\Delta t}$
- Substitute a into the equation for $\dfrac{\Delta v}{\Delta t}$. $\quad \dfrac{v}{r} = \dfrac{a}{v}$
- Multiply both sides of the equation by v. $\quad a = \dfrac{v^2}{r}$

The magnitude of the acceleration of an object moving with uniform circular motion is $a = v^2/r$. To determine its direction, again inspect the triangle formed by the velocity vectors in Figure 2.6. The acceleration is changing constantly, so imagine a vector $\vec{v}_2$ as close to $\vec{v}_1$ as possible. The angle θ is extremely small. In this case, $\Delta\vec{v}$ is almost exactly perpendicular to both $\vec{v}_1$ and $\vec{v}_2$. Since $\vec{v}_1$ and $\vec{v}_2$ are tangent to the circle and therefore are perpendicular to the associated radii of the circle, the acceleration vector points directly toward the centre of the circle.

Describing the acceleration vector in a typical Cartesian coordinate system would be extremely difficult, because the direction is always changing and, therefore, the magnitude of the *x*- and *y*-components would always be changing. It is much simpler to specify only the magnitude of the acceleration, which is constant for uniform circular motion, and to note that the direction is always toward the centre of the circle. To indicate this, physicists speak of a "centre-seeking acceleration" or **centripetal acceleration**, which is denoted as a_c, without a vector notation.

CENTRIPETAL ACCELERATION

Centripetal acceleration is the quotient of the square of the velocity and the radius of the circle.

$$a_c = \frac{v^2}{r}$$

Quantity	Symbol	SI unit
centripetal acceleration	a_c	$\frac{\text{m}}{\text{s}^2}$ (metres per second squared)
velocity (magnitude)	v	$\frac{\text{m}}{\text{s}}$ (metres per second)
radius (of circle)	r	m (metres)

Unit Analysis

$$\frac{\text{metre}}{\text{second}^2} = \frac{\left(\frac{\text{metre}}{\text{second}}\right)^2}{\text{metre}} \qquad \frac{\left(\frac{\text{m}}{\text{s}}\right)^2}{\text{m}} = \frac{\frac{\text{m}^{\cancel{2}}}{\text{s}^2}}{\cancel{\text{m}}} = \frac{\text{m}}{\text{s}^2}$$

Note: The direction of the centripetal acceleration is always along a radius pointing toward the centre of the circle.

Centripetal Force

According to Newton's laws of motion, an object will accelerate only if a force is exerted on it. Since an object moving with uniform circular motion is always accelerating, there must always be a force exerted on it in the same direction as the acceleration, as illustrated in Figure 2.7. If at any instant the force is withdrawn, the object will stop moving along the circular path and will proceed to move with uniform motion, that is, in a straight line that is tangent to the circular path on which it had been moving.

Since the force causing a centripetal acceleration is always pointing toward the centre of the circular path, it is called a **centripetal force**. The concept of centripetal force differs greatly from that of other forces that you have encountered. It is not a type of force such as friction or gravity. It is, instead, a force that is *required* in order for an object to move in a circular path.

A centripetal force can be supplied by any type of force. For example, as illustrated in Figure 2.8, gravity provides the centripetal force that keeps the Moon on a roughly circular path around Earth, friction provides a centripetal force that causes a car to move in a circular path on a flat road, and the tension in a string tied to a ball will cause the ball to move in a circular path when you twirl it around. In fact, two different types of force could act together to provide a centripetal force.

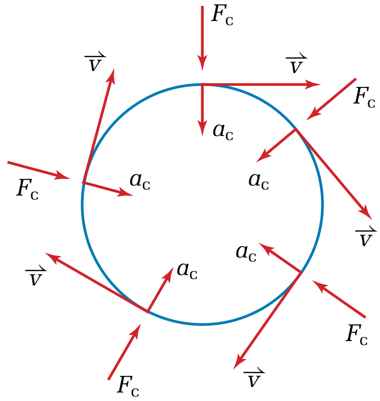


Figure 2.7 A force acting perpendicular to the direction of the velocity is always required in order for any object to move continuously along a circular path.

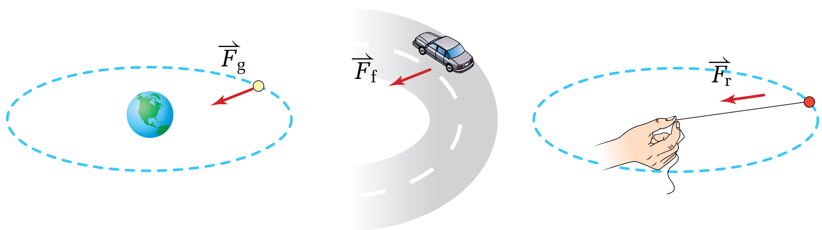


Figure 2.8 Any force that is directed toward the centre of a circle can provide a centripetal force.

You can determine the magnitude of a centripetal force required to cause an object to travel in a circular path by applying Newton's second law to a mass moving with a centripetal acceleration.

- Write Newton's second law. $\vec{F} = m\vec{a}$
- Write the equation describing centripetal acceleration. $a_c = \dfrac{v^2}{r}$
- Substitute into Newton's second law. Omit vector notations because the force and acceleration always point toward the centre of the circular path. $F_c = \dfrac{mv^2}{r}$

The equation for the centripetal force required to cause a mass m moving with a velocity v to follow a circular path of radius r is summarized in the following box.

CENTRIPETAL FORCE

The magnitude of the centripetal force is the quotient of the mass times the square of the velocity and the radius of the circle.

$$F_c = \frac{mv^2}{r}$$

Quantity	**Symbol**	**SI unit**
centripetal force	F_c	N (newtons)
mass	m	kg (kilograms)
velocity	v	$\frac{m}{s}$ (metres per second)
radius of circular path	r	m (metres)

Unit Analysis

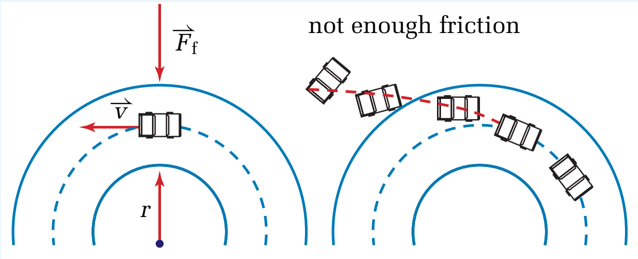
$$(\text{newtons}) = \left(\frac{\text{kilogram}\left(\frac{\text{metres}}{\text{second}}\right)^2}{\text{metres}}\right)$$

$$N = \frac{\text{kg}\left(\frac{\text{m}}{\text{s}}\right)^2}{\text{m}} = \frac{\text{kg}\frac{\text{m}^{\cancel{2}}}{\text{s}^2}}{\cancel{\text{m}}} = \frac{\text{kg}\cdot\text{m}}{\text{s}^2} = \text{N}$$

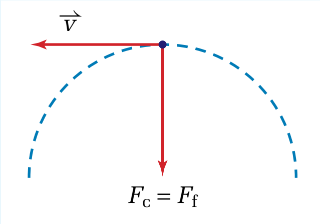
SAMPLE PROBLEMS

Centripetal Force in a Horizontal and a Vertical Plane

1. A car with a mass of 2135 kg is rounding a curve on a level road. If the radius of curvature of the road is 52 m and the coefficient of friction between the tires and the road is 0.70, what is the maximum speed at which the car can make the curve without skidding off the road?



Conceptualize the Problem



- Make a sketch of the motion of the car and the forces acting on it.
- The *force of friction* must provide a sufficient *centripetal force* to cause the car to follow the curved road.
- The magnitude of *force* required to keep the car on the road depends on the *velocity* of the car, its *mass*, and the *radius of curvature* of the road.
- Since r is in the denominator of the expression for centripetal force, as the *radius* becomes *smaller*, the amount of *force* required becomes *greater*.
- Since v is in the numerator, as the *velocity* becomes *larger*, the *force* required to keep the car on the road becomes *greater*.

Identify the Goal

The maximum speed, v, at which the car can make the turn

Identify the Variables

Known		Implied	Unknown	
$m = 2135$ kg	$\mu = 0.70$	$g = 9.81\ \frac{\text{m}}{\text{s}^2}$	F_f	F_N
$r = 52$ m			v	

Develop a Strategy

Set the frictional force equal to the centripetal force.

$$F_f = F_c$$
$$\mu F_N = \frac{mv^2}{r}$$

Since the car is moving on a level road, the normal force of the road is equal to the weight of the car. Substitute mg for F_N.

$$\mu mg = \frac{mv^2}{r}$$

Solve for the velocity.

$$v^2 = \mu \cancel{m} g\left(\frac{r}{\cancel{m}}\right)$$
$$v = \sqrt{\mu rg}$$

Substitute in the numerical values and solve.

$$v = \sqrt{(0.70)(52\ \text{m})\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)}$$
$$v = \sqrt{357.08\ \frac{\text{m}^2}{\text{s}^2}}$$
$$v = 18.897\ \frac{\text{m}}{\text{s}}$$
$$v \cong 19\ \frac{\text{m}}{\text{s}}$$

If the car is going faster than 19 m/s, it will skid off the road.

Validate the Solution

A radius of curvature of 52 m is a sharp curve. A speed of 19 m/s is equivalent to 68 km/h, which is a high speed at which to take a sharp curve. The answer is reasonable. The units cancelled properly to give metres per second for velocity.

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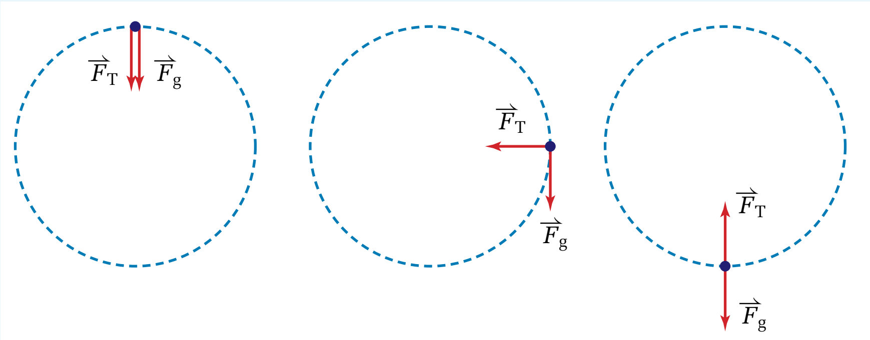
2. **You are playing with a yo-yo with a mass of 225 g. The full length of the string is 1.2 m. You decide to see how slowly you can swing it in a vertical circle while keeping the string fully extended, even when the yo-yo is at the top of its swing.**

(a) **Calculate the minimum speed at which you can swing the yo-yo while keeping it on a circular path.**

(b) **At the speed that you determine in part (a), find the tension in the string when the yo-yo is at the side and at the bottom of its swing.**

Conceptualize the Problem

- Draw *free-body diagrams* of the yo-yo at the *top, bottom, and one side* of the swing.



- At the *top* of the swing, both *tension and the force of gravity* are acting *toward the centre* of the circle.
- If the required *centripetal force* is *less than the force of gravity*, the yo-yo will *fall* away from the circular path.
- If the required *centripetal force* is *greater than the force of gravity*, the *tension* in the string *will* have to *contribute* to the centripetal force.
- Therefore, the *smallest possible velocity* would be the case where the required *centripetal force* is exactly *equal* to the *force of gravity*.
- At the *side* of the swing, the force of *gravity is perpendicular* to the direction of the required centripetal force and therefore contributes *nothing*. The centripetal force must all be supplied by the tension in the string.
- At the *bottom* of the swing, the force of *gravity* is in the *opposite* direction from the required *centripetal force*. Therefore, the *tension* in the string must *balance* the force of *gravity* and *supply* the required *centripetal force*.

Identify the Goal

The minimum speed, v, at which the yo-yo will stay on a circular path
The tension, F_T, in the string when the yo-yo is at the side of its circular path
The tension, F_T, in the string when the yo-yo is at the bottom of its circular path

Identify the Variables

Known	Implied	Unknown
$m = 225$ kg	$g = 9.81\ \frac{\text{m}}{\text{s}^2}$	v_{min}
$r = 1.2$ m		$F_{\text{T(side)}}$
		$F_{\text{T(bottom)}}$

Develop a Strategy

Set the force of gravity on the yo-yo equal to the centripetal force and solve for the velocity.

$$F_g = F_c$$
$$mg = \frac{mv^2}{r}$$
$$\cancel{m}g\left(\frac{r}{\cancel{m}}\right) = v^2$$
$$v = \sqrt{gr}$$

Substitute numerical values and solve.

$$v = \sqrt{\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)(1.2\ \text{m})}$$
$$v = \sqrt{11.772\ \frac{\text{m}^2}{\text{s}^2}}$$

A negative answer has no meaning in this application.

$$v = \pm 3.431\ \frac{\text{m}}{\text{s}}$$
$$v \cong 3.4\ \frac{\text{m}}{\text{s}}$$

(a) The minimum speed at which the yo-yo can move is 3.4 m/s.

Set the force of tension in the string equal to the centripetal force. Insert numerical values and solve.

$$F_T = F_c$$
$$F_T = \frac{mv^2}{r}$$
$$F_T = \frac{(225\ \cancel{\text{g}})\left(\frac{1\ \text{kg}}{1000\ \cancel{\text{g}}}\right)\left(3.431\ \frac{\text{m}}{\text{s}}\right)^2}{1.2\ \text{m}}$$
$$F_T = 2.207\ \frac{\text{kg}\frac{\text{m}^{\cancel{2}}}{\text{s}^2}}{\cancel{\text{m}}}$$
$$F_T \cong 2.2\ \text{N}$$

(b): Side – When the yo-yo is at the side of its swing, the tension in the string is 2.2 N.

Set the centripetal force equal to the vector sum of the force of tension in the string and the gravitational force. Solve for the force due to the tension in the string.

$$F_c = F_T + F_g$$
$$\frac{mv^2}{r} = F_T - mg$$
$$F_T = \frac{mv^2}{r} + mg$$

Substitute numerical values and solve.

$$F_T = \frac{(225\ \cancel{\text{g}})\left(\frac{1\ \text{kg}}{1000\ \cancel{\text{g}}}\right)\left(3.431\ \frac{\text{m}}{\text{s}}\right)^2}{1.2\ \text{m}} + (225\ \cancel{\text{g}})\left(\frac{1\ \text{kg}}{1000\ \cancel{\text{g}}}\right)\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)$$
$$F_T = 2.207\ \frac{\frac{\text{kg}\cdot\text{m}^{\cancel{2}}}{\text{s}^2}}{\cancel{\text{m}}} + 2.207\ \frac{\text{kg}\cdot\text{m}}{\text{s}^2}$$
$$F_T = 4.414\ \text{N}$$
$$F_T \cong 4.4\ \text{N}$$

(b): Bottom – When the yo-yo is at the bottom of its swing, the tension in the string is 4.4 N.

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Validate the Solution

The force of gravity (weight) of the yo-yo is 2.2 N. At the top of the swing, the weight supplies the entire centripetal force and the speed of the yo-yo is determined by this value. At the side of the swing, the tension must provide the centripetal force and the problem was set up so that the centripetal force had to be equal to the weight of the yo-yo, or 2.2 N. At the bottom of the swing, the tension must support the weight (2.2 N) and, in addition, provide the required centripetal force (2.2 N). You would therefore expect that the tension would be twice the weight of the yo-yo. The units cancel properly to give newtons for force.

PRACTICE PROBLEMS

15. A boy is twirling a 155 g ball on a 1.65 m string in a horizontal circle. The string will break if the tension reaches 208 N. What is the maximum speed at which the ball can move without breaking the string?

16. An electron (mass 9.11×10^{-31} kg) orbits a hydrogen nucleus at a radius of 5.3×10^{-11} m at a speed of 2.2×10^{6} m/s. Find the centripetal force acting on the electron. What type of force supplies the centripetal force?

17. A stone of mass 284 g is twirled at a constant speed of 12.4 m/s in a vertical circle of radius 0.850 m. Find the tension in the string **(a)** at the top and **(b)** at the bottom of the revolution. **(c)** What is the maximum speed the stone can have if the string will break when the tension reaches 33.7 N?

18. You are driving a 1654 kg car on a level road surface and start to round a curve at 77 km/h. If the radius of curvature is 129 m, what must be the frictional force between the tires and the road so that you can safely make the turn?

19. A stunt driver for a movie needs to make a 2545 kg car begin to skid on a large, flat, parking lot surface. The force of friction between his tires and the concrete surface is 1.75×10^{4} N and he is driving at a speed of 24 m/s. As he turns more and more sharply, what radius of curvature will he reach when the car just begins to skid?

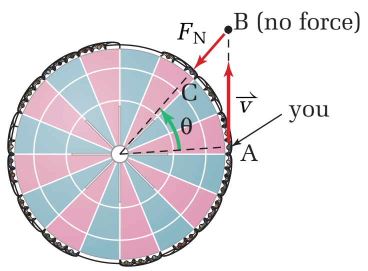

Figure 2.9 Assume that the Round Up ride is rotating at a constant speed and you are at point A. After a short time interval, in the *absence* of a force acting on you, you would move to point B, radially outward from point C. A centripetal force is required to change the direction of your velocity and place you at point C.

Centripetal Force versus Centrifugal Force

You read in Chapter 1, Fundamentals of Dynamics, that a centrifugal force is a fictitious force. Now that you have learned about centripetal forces, you can understand more clearly why a centrifugal force is classed as fictitious.

Analyze the motion of and the force on a person who is riding the Round Up. Imagine that Figure 2.9 is a view of the Round Up ride from above and at some instant you are at point A on the ride. At that moment, your velocity ($\vec{v}$) is tangent to the path of the ride. If no force was acting on you at all, you would soon be located at point B. However, the solid cylindrical structure of the ride exerts a normal force on you, pushing you to point C. There is no force pushing you outward, just a centripetal force pushing you toward the centre of the circular ride.